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The Holevo Cramér-Rao bound for multi-parameter quantum metrology

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Abstract

The Holevo Cramér-Rao bound is the most fundamental precision bound for multi-parameter quantum estimation. We show that for finite-dimensional systems it can be computed by solving a semidefinite program and we discuss how to implement this numerical optimization efficiently, by taking into account the rank of the state. This bound is more informative for various physically relevant quantum metrological problems. In particular, we consider two physical applications, phase and loss estimation in optical interferometry with fixed total photon number states and three-dimensional magnetometry with multiple qubits.

Meaningful precision bounds for multi-parameter quantum estimation are hard to obtain, due to the non-commutativity of observables in quantum mechanics. The Holevo Cramér Rao bound [1, 2] (HCRB) represents the ultimate lower bound for the sum of the variances of the estimates of the parameters. In the asymptotic limit of an infinite number of copies of the state, the HCRB is always attainable by implementing collective measurements on all the copies [3]; by restricting to single-copy measurements it is attainable for all pure states [4], but not in general. Crucially, the HCRB is always more informative than the bounds obtained from symmetric and right logarithmic derivatives (SLD and RLD).

For specific classes of states, the HCRB coincides with the SLD and RLD bounds [5], but a generic closed form is not known and it is thus expressed implicitly as a minimization. We show that such a minimization is the minimization of a convex function and that it can easily be converted into a semi-definite program (SDP), for which many efficient numerical methods have been developed in the last decades (we restrict to finite-dimensional systems). We then present an efficient numerical implementation that takes into account the rank of the state to reduce the complexity of the SDP.

We argue that there are several interesting problems in quantum metrology for which the HCRB should be considered to have more informative results: we study two well-known problems as examples. The first problem is joint estimation of phase and loss in optical interferometry by using fixed total photon number states [6]. The second example is sensing of a three-dimensional magnetic field with multi-qubit states [7], we consider both the noiseless and noisy scenarios.

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